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A Characterization for the Gevrey–Sobolev Wave Front Set

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Abstract In this note, we use the so-called microlocal energy method to give a characterization of the Gevrey–Sobolev wave front set $WF_{H^s_{\tau,\sigma}}(u)$, which will be useful in the study of non-linear microlocal analysis in Gevrey classes.

Keywords Gevrey–Sobolev space, Wave front set, Microlocal energy method
1991MR Subject Classification 35P20, 35P25

1 Introduction and Main Results

As is well-known, the microlocal energy method has had a fairly wide range of applications in the microlocal analysis, particularly in the Gevrey classes, for the linear partial differential equations (cf. [1]–[4]). However, how to extend this method to the nonlinear microlocal analysis, this is a pressing problem at present. We know the fundamental point in microlocal analysis is to depict the wave front set, and usually if we study the nonlinear PDE we need to consider the problem in Sobolev spaces. So, as a starting point, we need to study the wave front set in Sobolev spaces (or called the Sobolev wave front set). In this note, we shall consider the problem in the Gevrey classes and try to use the microlocal energy method to study the Gevrey–Sobolev wave front set. More precisely, we shall give a characterization for the Gevrey–Sobolev wave front set, which will be useful for us in the study of singularity analysis for non linear PDE in Gevrey classes (e.g. as a application of this method, the propagation of Gevrey singularities for nonlinear PDE will be studied in a forthcoming paper).

We know (cf. [1]–[4]) the microlocal energy method in Gevrey classes G^σ ($\sigma > 1$) will depend on two kinds of cut-off functions in x-space and ξ-space separately, more precisely they

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will actually depend on a sequence of cut-off functions $\{\alpha_i(\xi), \beta_i(x)\}$, and the latter will be required to satisfy some special estimates.

Let $(x_0, \xi^0) \in T^*\mathbb{R}^n \backslash \{0\}$, $|\xi^0| = 1$, as a fixed point, we define $\alpha_i(\xi)$ and $\beta_i(x)$ as follows:

Definition of $\alpha_i(\xi)$ *Let* $\chi_N(\xi) \in C_0^\infty$, *depending on parameter* N, *with* supp $\chi_N \subset \{\xi | |\xi - \xi^0| \leq r_0\}$, $0 \leq \chi_N(\xi) \leq 1$, *and satisfy the following two conditions*

$$\chi_N(\xi) = 1, \text{ for } |\xi - \xi^0| \leq \frac{r_0}{2}, \tag{1.1}$$

$$|\partial_\xi^{\mu+\nu} \chi_N(\xi)| = |\chi_N^{(\mu+\nu)}(\xi)| \leq (Ncr_0^{-1})^{|\mu|} \nu!^{1+\varepsilon'} (cr_0^{-1})^{|\nu|}, \text{ for } |\mu| \leq N,$$

where $r_0 > 0$, $\varepsilon' > 0$, $1 + 2\varepsilon' < \sigma$, *and* c *is an absolute constant* (*i.e. independent of* r_0). *We remark that* $\mu, \nu \in Z_+^n$, *and* ν *can be taken arbitrarily, but* μ *is restricted by* $|\mu| \leq N$. *The construction of the cut-off function* $\chi_N(\xi)$ *is rather delicate, we may here refer to Hörmander* [5] *and Rodino* [6]. *Assume*

$$\alpha_i(\xi) = \chi_N\Big(\frac{\xi}{i}\Big), \quad i \in Z_+, \tag{1.2}$$

then

$$|\alpha_i^{(\mu+\nu)}(\xi)| \leq \Big(\frac{N}{i} cr_0^{-1}\Big)^{|\mu|} \nu!^{1+\varepsilon'} (cr_0^{-1})^{|\nu|} i^{-|\nu|}, \text{ for } |\mu| \leq N. \tag{1.3}$$

Here i *and* N *are related by:* N *is the interger nearest to* $(ce)^{-1} r_0 i^{\frac{1}{\sigma}}$.

Similarly we have:

Definition of $\beta_i(x)$ *Let* $\beta_i(x) \in C_0^\infty$, $0 \leq \beta_i(x) \leq 1$, *supported in* $|x - x_0| \leq r_0$, *and* $\beta_i(x) = 1$ *for* $|x - x_0| \leq \frac{r_0}{2}$, *satisfy*

$$|D_x^{(\mu+\nu)} \beta_i(x)| = |\beta_{i(\mu+\nu)}(x)| \leq (Ncr_0^{-1})^{|\mu|} \nu!^{1+\varepsilon'} (cr_0^{-1})^{|\nu|}, \text{ for } |\mu| \leq N. \tag{1.4}$$

We also call $\{\alpha_i(D_x), \beta_i(x)\}$ the microlocalizer around the point (x_0, ξ^0) with size r_0. The conditions above on $\alpha_i(\xi)$ and $\beta_i(x)$ enable us to use two different kinds of estimates, this makes our treatment fairly easy in the case of the Gevrey class. In other words, by using the microlocalizer $\{\alpha_i(D_x), \beta_i(x)\}$, we can deal with the problem from the microlocal view-point. This method is called the microlocal energy method by Mizohata [1]–[3], where he also gave a characterization of the Gevrey wave front set $WF_\sigma(u)$ for a σ-ultradistribution u.

Proposition 1.1 (see [2]) *Let* u *be a* σ*-ultradistribution* ($\sigma > 1$), *and* $(x_0, \xi^0) \notin WF_\sigma(u)$ (*with* $|\xi^0| = 1$). *Then, if* r_0 *is small enough and* i *is large enough, we have* $S_i = \sum_{|\mu+\nu| \leq N} C_{\mu\nu}^i \|\alpha_i^{(\mu)} \beta_{i(\nu)} u\| \leq \exp\{-\varepsilon i^{\frac{1}{\sigma}}\}$, *where* $C_{\mu\nu}^i = i^{(1-\frac{1}{\sigma})|\mu| - \frac{1}{\sigma}|\nu|}$, *and* ε *is a positive constant which could be chosen independent of* r_0 *when* r_0 *tends to* 0.

The converse of Proposition 1.1 also holds. Actually we can state it in a strong form.

Proposition 1.2 (see [2]) *Let* u *be a* σ*-ultradistribution* ($\sigma > 1$). *If we denote* $C_{\mu 0}^i = i^{(1-\frac{1}{\sigma})|\mu|}$ *then an estimate of the form* $\widetilde{S}_i = \sum_{|\mu| \leq N} C_{\mu 0}^i \|\alpha_i^{(\mu)} \beta_i u\| \leq \exp\{-\varepsilon_0 i^{\frac{1}{\sigma}}\}, (\exists \varepsilon_0 > 0)$, *for large* i, *implies* $(x_0, \xi^0) \notin WF_\sigma(u)$.

In this paper we shall extend the results in Propositions 1.1 and 1.2 to the case of the Gevrey–Sobolev wave front set $WF_{H^s_{\tau,\sigma}}(u)$, i.e. by using the microlocal energy method, to give a characterization of $WF_{H^s_{\tau,\sigma}}(u)$. Let us first give the definition of the Gevrey–Sobolev space.

Suppose $\sigma > 1,\ \tau,\ s \in \mathbb{R}$. The Gevrey–Sobolev spaces are defined by

$$H^s_{\tau,\sigma}(\mathbb{R}^n) = \{u \in \mathcal{S}'_{-\tau,\sigma}(\mathbb{R}^n), \exp[\tau\langle D\rangle^{1/\sigma}]u \in H^s(\mathbb{R}^n)\}, \tag{1.5}$$

where $\langle D\rangle = (1-\Delta)^{1/2}$, the space $\mathcal{S}'_{\tau,\sigma}$ is defined as the dual space of $\mathcal{S}_{\tau,\sigma}$ which in turn, for $\tau \geq 0$, is defined by the inverse Fourier transform from $\widehat{\mathcal{S}}_{\tau,\sigma} = \{v(\xi) \in C^\infty(\mathbb{R}^n) \mid \exp[\tau\langle \xi\rangle^{1/\sigma}]v(\xi) \in \mathcal{S}(\mathbb{R}^n)\}$; for $\tau < 0$, the space $\mathcal{S}_{\tau,\sigma}$ is defined by the transposition of the inverse Fourier transform from $\widehat{\mathcal{S}}_{\tau,\sigma}$ (cf. [7]). The infinite order pseudo-differential operator $\exp[\tau\langle D\rangle^{1/\sigma}]$ is defined by the Fourier transform as usual (see [6]).

We know $H^s_{\tau,\sigma}$ is a Hilbert space with inner product

$$\langle u, v\rangle_{H^s_{\tau,\sigma}} = \langle \exp[\tau\langle D\rangle^{1/\sigma}]u, \exp[\tau\langle D\rangle^{1/\sigma}]v\rangle_{H^s}, \tag{1.6}$$

and the norm is defined by

$$\|u\|_{H^s_{\tau,\sigma}} = \|\exp[\tau\langle D\rangle^{1/\sigma}]u\|_{H^s}. \tag{1.7}$$

Next we denote $H^s_{\tau,\sigma,\mathrm{loc}}$ as the Gevrey locally Sobolev spaces, i.e. $u \in H^s_{\tau,\sigma,\mathrm{loc}}$ means that u is a σ-ultradistribution and for every $\phi \in G_0^{\sigma'}(\mathbb{R}^n)$ with $1 < \sigma' < \sigma$ we have $\phi u \in H^s_{\tau,\sigma}$. If we write V_{x_0} for a neighborhood of $x_0 \in \mathbb{R}^n$, we say $u \in H^s_{\tau,\sigma}(x_0)$ if there exists V_{x_0} such that for all $\phi \in G_0^{\sigma'}(V_{x_0}),\ 1 < \sigma' < \sigma$, we have $\phi u \in H^s_{\tau,\sigma}$. Observe that $\bigcup_{s\in\mathbb{R},\tau>0} H^s_{\tau,\sigma}(x_0) = G^\sigma(x_0)$, the space of all the functions u which are of class G^σ in a neighborhood of x_0; moreover $G^{\sigma'}(x_0) \subset H^s_{\tau,\sigma}(x_0)$ with strict inclusion for all $s \in \mathbb{R},\ \tau > 0,\ 1 < \sigma' < \sigma$ (see [8]).

Let $(x_0, \xi^0) \in T^*\mathbb{R}^n\backslash\{0\}$, We say $u \in H^s_{\tau,\sigma}(x_0, \xi^0)$ ($\sigma > 1,\ s,\ \tau \in \mathbb{R}$), is a Gevrey microlocally (i.e. near (x_0, ξ^0)) Sobolev space, if there exist V_{x_0} and a conic neighborhood Γ_0 of ξ^0 in $\mathbb{R}^n\backslash\{0\}$, such that for all $\phi \in G_0^{\sigma'}(V_{x_0}),\ 1 < \sigma' < \sigma$, and every $\psi \in C^\infty(\mathbb{R}^n_\xi)$, 0-order homogeneous in ξ for large $|\xi|$ with supp $\psi \subset \Gamma_0$, we have $\psi(D_x)(\phi u) \in H^s_{\tau,\sigma}$. Thus we have:

Definition *Let u be a σ-ultradistribution, we say $(x_0, \xi^0) \notin WF_{H^s_{\tau,\sigma}}(u)$ if and only if $u \in H^s_{\tau,\sigma}(x_0, \xi^0)$.*

Observe that for $(x_0, \xi^0) \notin WF_{H^s_{\tau,\sigma}}(u)$, it is equivalent to say that there exist $\phi \in G_0^{\sigma'}(\mathbb{R}^n)$, $1 < \sigma' < \sigma$, with $\phi \equiv 1$ near x_0, and a conic neighborhood Γ_0 of ξ^0 in $\mathbb{R}^n\backslash\{0\}$ (in this paper we always assume $|\xi^0| = 1$, and for $r_0 > 0$, $\Gamma_0 = \Gamma(\xi^0, r_0) = \{\xi \in \mathbb{R}^n\backslash\{0\} |\ |\frac{\xi}{|\xi|} - \xi^0| < r_0\}$), such that

$$\int_{\Gamma_0} \exp[2\tau\langle\xi\rangle^{\frac{1}{\sigma}}]\langle\xi\rangle^{2s}|\hat{\phi u}(\xi)|^2 d\xi < \infty. \tag{1.8}$$

Now we shall give the main results of this note:

Theorem 1.1 *Let u be a σ-ultradistribution, $(x_0, \xi^0) \in T^*\mathbb{R}^n$ with $|\xi^0| = 1$. If $(x_0, \xi^0) \notin WF_{H^s_{\tau,\sigma}}(u)$, more precisely if there exist $\phi \in G_0^{\sigma'}(\mathbb{R}^n),\ 1 < \sigma' < \sigma,\ \phi \equiv 1$ on $\{|x - x_0| \leq \frac{r_0}{2}\}$ $(\exists r_0 > 0)$, $\phi = 0$ outside of $\{|x-x_0| < r_0\}$, and a conic neighborhood $\Gamma(\xi^0, r_0') = \{|\frac{\xi}{|\xi|} - \xi^0| <$*

$r'_0\}$ $(\exists r'_0 > 0)$ *of* ξ^0 *in* $\mathbb{R}^n\backslash\{0\}$*, such that*

$$\int_{\Gamma(\xi^0,r'_0)} \exp[2\tau\langle\xi\rangle^{\frac{1}{\sigma}}]\langle\xi\rangle^{2s}|\mathcal{F}(\phi u)(\xi)|^2 d\xi < \infty. \tag{1.9}$$

Then for any microlocalizer $\{\alpha_i(D_x), \beta_i(x)\}$ *around* (x_0, ξ^0) *with size* $r''_0 \le \frac{\min\{r_0, r'_0\}}{2}$*, we have*

$$\sum_i \|\alpha_i\beta_i u\|^2_{L^2_{\tau,\sigma}} i^{2s-1} = \sum_i \|\exp[\tau\langle D\rangle^{\frac{1}{\sigma}}]\alpha_i\beta_i u\|^2_{L^2} i^{2s-1} < \infty. \tag{1.10}$$

Moreover, we have

$$\sum_i \|\alpha_i^{(\mu)}\beta_{i(\nu)} u\|^2_{L^2_{\tau,\sigma}} i^{2s+2|\mu|-1} < \infty, \text{ for all } \mu,\ \nu \in Z^n_+. \tag{1.11}$$

Conversely, we have:

Theorem 1.2 *Assume that for some microlocalizer* $\{\alpha_i(D_x), \beta_i(x)\}$ *around* $(x_0, \xi^0) \in T^*(\mathbb{R}^n)$, $|\xi^0| = 1$ *with size* r_0*, it holds for a* σ*-ultradistribution* u

$$\sum_i \|\alpha_i\beta_i u\|^2_{L^2_{\tau,\sigma}} i^{2s-1} < \infty. \tag{1.12}$$

Then for any r'_0 *satisfying* $r'_0 < \frac{r_0}{2}$*, we have*

$$\int_{\Gamma(\xi^0,r'_0)} \exp[2\tau\langle\xi\rangle^{\frac{1}{\sigma}}]\langle\xi\rangle^{2s}|\mathcal{F}(\beta_i u)(\xi)|^2 d\xi < \infty, \tag{1.13}$$

which implies $(x_0, \xi^0) \notin WF_{H^s_{\tau,\sigma}}(u)$.

Remark The microlocal energy method can be also used to study a similar problem in C^∞ category.

2 Proof of Results

Proof of Theorem 1.1 First we prove that for any $i \in N$

$$\int_{\Gamma(\xi^0,r''_0)} \exp[2\tau\langle\xi\rangle^{\frac{1}{\sigma}}]\langle\xi\rangle^{2s}|\mathcal{F}(\beta_i u)(\xi)|^2 d\xi < \infty. \tag{2.1}$$

Let $\psi(\xi) \in C^\infty(\mathbb{R}^n)$ be essentially homogeneous of degree 0, with support contained in $\Gamma(\xi^0, r'_0)$, and $\psi(\xi) = 1$ for $\{\xi \mid |\frac{\xi}{|\xi|} - \xi^0| \le \frac{r'_0}{2}\} \bigcap \{\xi \mid |\xi| > A\}$ $(\exists A > 0)$. Then since $\phi \equiv 1$ on supp $\beta_i(x) \subset \{|x - x_0| < r''_0\}$, we have $\langle D\rangle^s\psi(D_x)\beta_i u = \langle D\rangle^s\psi(D_x)\beta_i(x)(\phi u)$. Commuting $\langle D\rangle^s\psi(D_x)$ with $\beta_i(x)$, we have

$$\langle D\rangle^s\psi(D_x)\beta_i u = \sum_{|\nu|<N} \nu!^{-1}\beta_{i(\nu)}(x)((\langle D\rangle^s\psi(D_x))^{(\nu)}(\phi u) + R_N(x, D_x)(\phi u),$$

where $R_N(x,\xi) \in S^{s-N}_{\sigma'}(\mathbb{R}^n)$, the $G^{\sigma'}$ pseudodifferential operator for some $\sigma' \in (1,\sigma)$; and we observe

$$\|\beta_{i(\nu)}(x)(\langle D\rangle^s \psi(D_x))^{(\nu)}(\phi u)\|^2_{L^2_{\tau,\sigma}} \leq c_\nu \int_{\mathrm{supp}[\psi]} \exp[2\tau\langle\xi\rangle^{\frac{1}{\sigma}}]\langle\xi\rangle^{2s}|\mathcal{F}(\phi u)(\xi)|^2 d\xi, \tag{2.2}$$

which is finite from (1.9). Next we know (cf. [8, Corollary 1.1]) that $R_N(x,D_x)$ is a continuous mapping from $H^{s'}_{\tau,\sigma}$ to $H^{s'-(s-N)}_{\tau,\sigma}$ for any $s' \in \mathbb{R}$; and we can assume that $\phi u \in H^{-k}_{\tau,\sigma}$ for some $k \geq 0$, thus we take N large enough so that $s - N \leq -k$, then

$$\|R_N(x,D_x)(\phi u)\|_{L^2_{\tau,\sigma}} \leq \text{const.}\ \|\phi u\|_{H^{-k}_{\tau,\sigma}} < \infty. \tag{2.3}$$

Thus the estimates (1.9), (2.2) and (2.3) give that the estimate (2.1) holds since $\psi(\xi) \equiv 1$ in $\Gamma(\xi^0, r_0'')$.

Secondly, we denote

$$h_s(\xi) = \sum_i \alpha_i(\xi)^2 i^{2s-1}, \tag{2.4}$$

where we know that $\{\alpha_i(D_x), \beta_i(x)\}$ is a microlocalizer around (x_0,ξ^0) with size r_0'', and then

$$\sum_i \|\alpha_i\beta_i u\|^2_{L^2_{\tau,\sigma}} i^{2s-1} = \int h_s(\xi)\exp[2\tau\langle\xi\rangle^{\frac{1}{\sigma}}]|\mathcal{F}(\beta_i u)(\xi)|^2 d\xi. \tag{2.5}$$

Since supp $h_s(\xi) \subset \Gamma(\xi^0, r_0'')$, so from (2.5) and (2.1), the estimate (1.10) holds if $h_s(\xi) \leq c\langle\xi\rangle^{2s}$ for a positive constant c which is independent of ξ. In fact let $\xi \in$ supp h_s be fixed, if supp α_i contains ξ, then $|\xi - i\xi^0| \leq ir_0''$, i.e. $i(1-r_0'') \leq |\xi| \leq i(1+r_0'')$, or equivalently $(1+r_0'')^{-1}|\xi| \leq i \leq (1-r_0'')^{-1}|\xi|$. Thus we obtain, if $r_0'' < \frac{1}{2}$, that

(1) $i^{2s-1} \leq$ const. $|\xi|^{2s-1}$ if $\xi \in$ supp α_i;

(2) the number of i, in which $\xi \in$ supp α_i, is less or equal to $2|\xi|$.

These imply

$$h_s(\xi) = \sum_i \alpha_i(\xi)^2 i^{2s-1} \leq c\cdot|\xi|\cdot|\xi|^{2s-1} \leq c\langle\xi\rangle^{2s}, \tag{2.6}$$

where $c > 0$ is independent of ξ. This proves the estimate (1.10) holds.

Next, since $\|\alpha_i^{(\mu)}(\xi)\| \leq$ const. $i^{-|\mu|}$ for any $\mu \in Z^n_+$, thus it is obvious that for any $\mu,\ \nu \in Z^n_+$ we have

$$\sum_i \|\alpha_i^{(\mu)}\beta_{i(\nu)}u\|^2_{L^2_{\tau,\sigma}} i^{2s+2|\mu|-1} < \infty. \tag{2.7}$$

This proves the estimate (1.11).

Proof of Theorem 1.2 Let $\xi \in \Gamma(\xi^0, r_0')$, $r_0' < \frac{r_0}{2}$. Then

$$\sum_i \|\alpha_i\beta_i u\|^2_{L^2_{\tau,\sigma}} i^{2s-1} = \int h_s(\xi)\exp[2\tau\langle\xi\rangle^{\frac{1}{\sigma}}]|\mathcal{F}(\beta_i u)(\xi)|^2 d\xi,$$

where $h_s(\xi) = \sum_i \alpha_i(\xi)^2 i^{2s-1}$. We need only to prove that

$$h_s(\xi) \geq c\cdot|\xi|^{2s}, \quad \text{for } \xi \in \Gamma(\xi^0, r_0'), \tag{2.8}$$

where $c > 0$ is independent of ξ.

In fact, here we introduce a positive constant ε_0 such that $2r_0'\varepsilon_0 = \frac{r_0}{2} - r_0'$, or equivalently $r_0'(1+2\varepsilon_0) = \frac{r_0}{2}$. Without loss of generality we can assume here $r_0 < 1$. For fixed $\xi \in \Gamma(\xi^0, r_0')$, then let i satisfy

$$|i - |\xi|| \leq \varepsilon_0 r_0' i, \tag{2.9}$$

which implies

$$|\xi - i\xi^0| = |\xi - |\xi|\xi^0 + |\xi|\xi^0 - i\xi^0| \leq |\xi - |\xi|\xi^0| + ||\xi| - i| \leq r_0'|\xi| + \varepsilon_0 r_0' i. \tag{2.10}$$

Also the estimate (2.9) is equivalent to

$$(1 - \varepsilon_0 r_0')i \leq |\xi| \leq (1 + \varepsilon_0 r_0')i, \quad \text{or } (1 + \varepsilon_0 r_0')^{-1}|\xi| \leq i \leq (1 - \varepsilon_0 r_0')^{-1}|\xi|. \tag{2.11}$$

Thus the estimates (2.10) and (2.11) give

$$|\xi - i\xi^0| \leq \frac{r_0}{2} i, \quad \text{i.e. } \alpha_i(\xi) = 1. \tag{2.12}$$

From the process above, we have that for $\xi \in \Gamma(\xi^0, r_0')$:

(1) the number of such i, for which the estimate (2.9) is to be satisfied (which also implies $\alpha_i(\xi) = 1$), can be estimated from below by $c_1|\xi|$ (where $c_1 = (1 - \varepsilon_0 r_0')^{-1} - (1 + \varepsilon_0 r_0')^{-1} > 0$).

(2) for such i as above, we have $i^{2s-1} \geq c_2|\xi|^{2s-1}$ ($\exists c_2 > 0$, independent of ξ).

Hence we have deduced that for $\xi \in \Gamma(\xi^0, r_0')$, $h_s(\xi) \geq \sum_{i;\alpha_i(\xi)=1} i^{2s-1} \geq c_1|\xi| i^{2s-1} \geq c_1 c_2 |\xi|^{2s}$. This implies (2.8). Theorem 1.2 is proved.

The following is an obvious corollary of Theorem 1.2:

Corollary 2.1 *Let u be a σ-ultradistribution, $(x_0, \xi^0) \in T^*\mathbb{R}^n \backslash \{0\}$. If $\{\alpha_i(D_x), \beta_i(x)\}$ is a microlocalizer around (x_0, ξ^0) with size r_0, and $\|\alpha_i \beta_i u\|_{L^2_{\tau,\sigma}} \in O(i^{-s})$, then for any $\varepsilon > 0$, we have $(x_0, \xi^0) \notin WF_{H^{s-\varepsilon}_{\tau,\sigma}}(u)$.*

Proof of Corollary 2.1 Since $\|\alpha_i \beta_i u\|^2_{L^2_{\tau,\sigma}} i^{2s} \leq C$, which means for any $\varepsilon > 0$ that $\sum_i \|\alpha_i \beta_i u\|^2_{L^2_{\tau,\sigma}} i^{2(s-\varepsilon)-1} \leq C \sum_i i^{-(1+2\varepsilon)} < \infty$. Thus from Theorem 1.2, we have $(x_0, \xi^0) \notin WF_{H^{s-\varepsilon}_{\tau,\sigma}}(u)$.

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